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Crack paths in three-dimensional elastic solids. I: two-term expansion of the stress intensity factors—application to crack path stability in hydraulic fracturing

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Abstract

The aim of this work is to lay theoretical foundations for the prediction of crack paths within the theory of quasistatic LEFM under the most general hypotheses: arbitrary three-dimensional geometry, arbitrary loading. This objective requires to derive the expression of the stress intensity factors along the crack front after an arbitrary infinitesimal propagation. Only the first two terms of their expansion in powers of the crack extension length δ, proportional to $\delta^0 = 1$ and $\delta^{1/2}$, are considered in this paper. Fully general formulae for these terms are obtained by combining arguments of dimensional analysis (scale changes) and regularity properties (continuity, differentiability) of the stresses at a fixed, given point with respect to δ for $\delta = 0$ derived from the Bueckner–Rice weight function theory. This notably allows to extend the Cotterell–Rice criterion for stable rectilinear propagation of a mode I crack under plane strain conditions to the three-dimensional case. As an application, a penny-shaped crack induced by hydraulic fracturing is considered. Conclusions concerning the influence of the orientation and depth of such a crack upon the stability of its coplanar propagation seem to be compatible with experimental evidence.

1. Introduction

The prediction of crack paths under general mixed-mode conditions, using the theory of quasistatic LEFM, has been the subject of many investigations in the *two-dimensional case* (plane strain conditions). These include both theoretical works devoted to infinitesimal extension from a given position (see e.g. Cotterell and Rice, 1980; Sumi et al., 1983; Leblond, 1989; Amestoy and Leblond, 1992), and numerical studies simulating propagation over arbitrarily long distances by step-by-step methods (see e.g. Sumi, 1986; Wawrzynek, 1987; Autesserre, 1995).

The (of course more difficult) study of the *three-dimensional case* began about a decade ago with the works of Rice (1985, 1989), Gao and Rice (1986, 1987a, b) and Gao (1988), devoted to the question of the stability of the fundamental (straight or circular) configuration of the crack front

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vs small perturbations *within the crack plane*, for semi-infinite, penny-shaped and external circular cracks in an infinite body. The treatment heavily relied on explicit knowledge of the crack-face weight functions for such geometries. All three modes were considered (except for the external circular crack) but the conclusions were mainly relevant in the case of pure mode I, the hypothesis of coplanar propagation being seldom verified in the presence of mode II or III. Also, Nazarov (1989) provided a similar treatment of the case of an infinite body containing a planar crack loaded in pure mode I but with an arbitrary contour. All these works were based on two essential elements: first, asymptotic expressions of the stress intensity factors (SIFs) along the front of an arbitrary infinitesimal extension exhibiting the influence of its length; second, use of some propagation criterion expressed in terms of these SIFs. Combination of these elements then led to prediction of the values of the extension length along the crack front.

Slight deviations from coplanarity were envisaged in the more recent works of Gao (1992), Xu et al. (1994) and Ball and Larralde (1995). Unfortunately, these three works yield conflicting results. It should also be noted that they do not rely on the same principle as the works quoted above; indeed they do not consider perturbations of the crack generated by adding some infinitesimal *crack extension*; the *whole* initially plane surface of the crack is perturbed, the small parameter in the perturbation analysis being the distance between the original and perturbed crack surfaces (as measured perpendicularly to the initial crack plane) instead of the length of the crack extension. Further comments on these works will be given in Part II of the present one.

The objective of the present work is to attack the problem under the most general possible hypotheses: three dimensions, arbitrary geometry of the body and the crack, arbitrary loading. (In fact, we shall be obliged to put certain restrictions on this very ambitious objective; these will be indicated in due time.) The principle of the treatment will differ from that of those works on nonplanar cracks just quoted and is more similar to that of earlier works on planar cracks, in that the perturbation of the crack will result from *addition of some small extension*: thus, just as in the works of Rice (1985, 1989), Gao and Rice (1986, 1987a, b), Gao (1988) and Nazarov (1989), the prediction of the crack path will result from combination of asymptotic expressions of the SIFs for infinitesimal crack extension lengths and some appropriate criterion. However the originality of the present paper will lie solely in the search for the expression of the SIFs along the front of the extended crack: indeed the formulation of the propagation criterion is really an open problem only in the presence of mode III and we shall only consider a mode I + II situation in the application presented at the end (the criterion used there will simply be the widely accepted "principle of local symmetry" of Goldstein and Salganik, 1974).

In spite of the similarity of principle with previous works on the case of a planar crack, a new approach will have to be used. Indeed the methods employed in these earlier works are inapplicable here, because of lack of explicit expressions of crack-face weight functions for arbitrary, nonplanar cracks. The technique which will be used is an extension of that devised by the author (Leblond, 1989) for the study of the two-dimensional case (the results of which were more recently confirmed by Leguillon (1993) using another method). It is based first on scale changes and dimensional analysis, second on regularity properties (continuity, differentiability) of the mechanical fields at a given, fixed point with respect to the crack extension length δ for $\delta = 0$; the proof of differentiability relies on Rice's (1985) formulation of the theory of Bueckner's weight functions but does not require any precise knowledge of these functions. The results consist in formulae which specify the general *functional form* of the successive terms of the expansion of the SIFs in powers of δ in terms of the various mechanical and geometric parameters, notably those characterizing the shape of the crack extension (length, kink angle, curvature parameters). The functions involved in these formulae can then be identified through analytical or numerical calculations carried out for some simple, special cases. Use of the criterion finally yields the values of the geometric parameters of the crack extension.

We shall restrict our attention in the present Part I to the first two terms of the expansion of the SIFs, proportional to $\delta^0 = 1$ and $\delta^{1/2}$, respectively. As will be seen, the expressions of these terms appear in fact as straightforward generalizations of those obtained in the two-dimensional case (Leblond, 1989; Leguillon, 1993); in particular they involve only the value of δ *at that point of the front where the SIFs are expressed.* (In contrast, the third term, proportional to $\delta^1 = \delta$, depends upon the values of δ along the whole crack front; this phenomenon was apparent in all previous works on planar cracks and also of course occurs for more general geometries, as will be detailed in Part II.)

Practical implications of the results are illustrated by extending Cotterell and Rice's (1980) stability analysis vs small out-of-plane perturbations, originally formulated for cracks propagating under plane strain conditions, to the three-dimensional case. As an application, a penny-shaped crack loaded in mode I by far stresses plus an internal pressure, as encountered in hydraulic fracturing, is considered. It is concluded that stability of plane propagation of such a crack vs small deviations from coplanarity depends on both its orientation and depth under the ground surface: crack orientations perpendicular to the major or intermediate (in absolute value) principal far stresses lead to instability, whereas cracks orthogonal to the minor (in absolute value) principal far stress propagate in a stable manner provided that they lie at a sufficient depth. These theoretical conclusions find some support in the fact that cracks induced by hydraulic fracturing are almost always observed in practice to develop perpendicularly to the direction of minimum compression, except sometimes in the case of very shallow depths.

2. Geometric description of the crack and asymptotic expression of the stress field

We consider, within a three-dimensional elastic body, a crack of arbitrary shape, except that both its surface $\mathscr{S}$ and front $\mathscr{F}$ are assumed to be of class $\mathscr{C}^\infty$, at least in the vicinity of $\mathscr{F}$ (this rules out angular points on $\mathscr{F}$ for instance) (Fig. 1). Let O denote an arbitrary point on $\mathscr{F}$. Cartesian coordinates x_1, x_2, x_3 are attached to that point, with Ox_1 in the tangent plane to $\mathscr{S}$ and orthogonal to $\mathscr{F}$, Ox_2 perpendicular to $\mathscr{S}$ and Ox_3 coincident with the tangent to $\mathscr{F}$. We shall note s the position (curvilinear length) of O on $\mathscr{F}$, and s' that of the generic point on that curve.

We wish to describe the local geometry of $\mathscr{S}$ with a degree of accuracy such that the distance x_2 from an arbitrary point on it to its projection onto the tangent plane at O be specified up to order $O(x_1^2+x_3^2)$. This is achieved by prescribing the components C_{11}, C_{13}, C_{33} of the curvature tensor $\mathbf{C}$ of $\mathscr{S}$ at the point O. The local geometry of $\mathscr{F}$ is then also described with a similar accuracy by prescribing the curvature Γ of its projection onto the tangent plane at O.

Let us now add a small arbitrary deviated extension to the crack (Fig. 2). We make the fundamental assumption that the original crack front lies in both the old and new crack surfaces, i.e. that *the crack extension develops continuously from that original front*; this rules out cases of largely predominant mode III, for which it is experimentally known (see e.g. Palaniswamy and

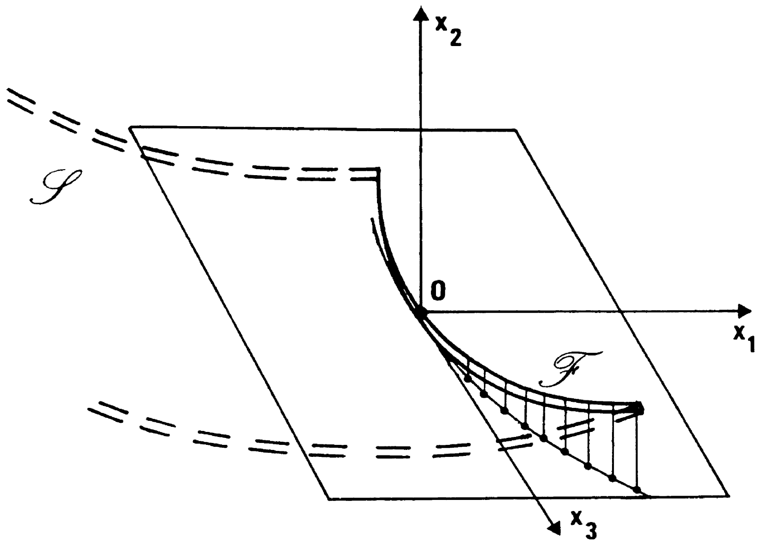

Fig. 1. Arbitrarily shaped crack in a three-dimensional body.

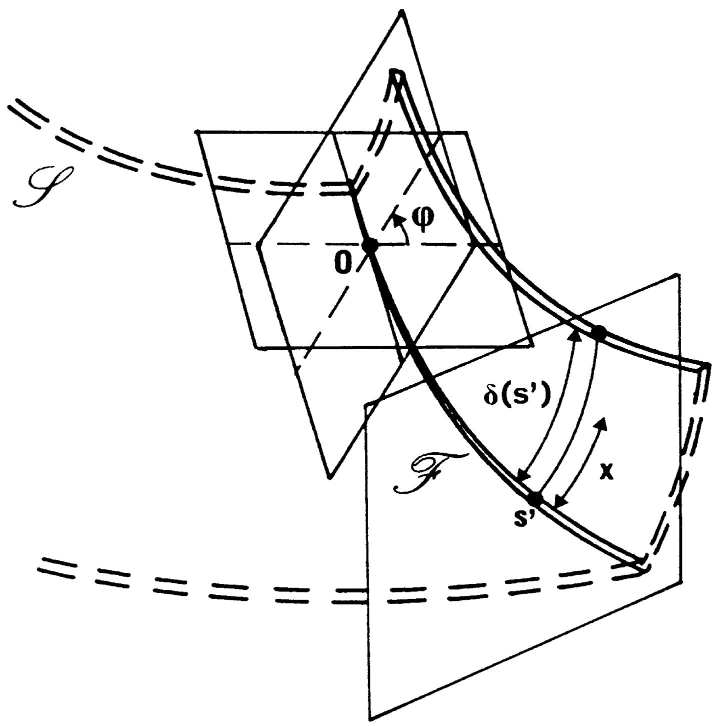

Fig. 2. Arbitrary crack with a small kinked and curved extension.

Knauss, 1975) that the new crack surface develops in the form of small, separate tilted ''pennies'' centred on the original crack front and intersecting the original crack surface roughly perpendicularly to that front. At each point s' of $\mathscr{F}$, the new tangent plane is obtained by rotating the old one about the local tangent to $\mathscr{F}$ by an angle $\varphi(s')$ (the *kink angle*; see Fig. 2); the value $\varphi(s)$ of the function $\varphi(s')$ at the point O will often be simply noted φ when no ambiguity will arise, and the same convention will be used for other functions defined on the crack front. The function $\varphi(s')$ will be assumed to be of class $\mathscr{C}^\infty$. (Some authors introduce another angle characterizing a *rotation of the tangent plane about the* x_1*-axis*. This angle will not be considered here, because this would violate the fundamental assumption mentioned above.)

We again wish to specify the distance from the crack extension to its tangent plane at the point O up to order 2 with respect to the distance to that point. Let us consider the intersection of the crack extension and the Ox_1x_2 plane (perpendicular to $\mathscr{F}$ at O). The shape of this intersection is assumed to be of the type described by the following expression:

$$x_2^* = a^* x_1^{*3/2} + \frac{C^*}{2} x_1^{*2} + O(x_1^{*5/2}); \quad x_3 = 0, \tag{1}$$

where Ox_1^*, Ox_2^* denote axes obtained by rotating Ox_1, Ox_2 about Ox_3 by the angle φ, and a^*, C^* parameters (Fig. 3). The necessity of introducing the term $a^* x_1^{*3/2}$ (resulting in an infinite curvature of the crack extension at the point O) to describe mixed mode propagation was established in many papers, e.g. Cotterell and Rice (1980) (for plane strain conditions, but the conclusion obviously remains valid in the more general three-dimensional case). It may then be verified that a local geometric description of the surface of the crack extension can be achieved with the desired degree of accuracy by specifying only (in addition to $\mathbf{C}$, Γ and φ) the parameters a^* and C^*, plus the derivative $\varphi' \equiv \mathrm{d}\varphi/\mathrm{d}s$ of the kink angle along $\mathscr{F}$ at the point O. This means that a^* being for instance assumed to be zero for simplicity, specifying $\mathbf{C}$, Γ, φ, C^* and φ' is sufficient to fix the curvature tensor of the crack extension at the point O.

The description of the crack extension *surface* must be completed by a description of the new crack *front*. Let $\delta(s')$ denote the curvilinear length of the crack extension, as measured

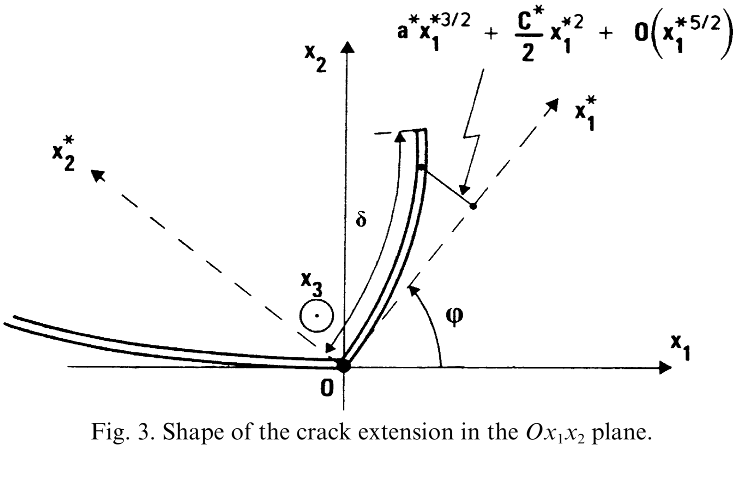

Fig. 3. Shape of the crack extension in the Ox_1x_2 plane.

perpendicularly to the original crack front (see Fig. 2). The function $\delta(s')$ will be assumed to be of class $\mathscr{C}^{\infty}$. The value $\delta \equiv \delta(s')$ of $\delta(s')$ at the point O will be assumed to be non-zero; this means that *O will be supposed not to be an endpoint of the "active", effectively propagating part of the front*. A local geometric description of the shape and position of the new crack front with respect to the old one can be achieved with the same degree of accuracy as before by specifying, in addition to the local extension length δ, its logarithmic derivative $\delta'/\delta \equiv \mathrm{d}\delta/(\delta\,\mathrm{d}s)$. This quantity is homogeneous to a curvature and represents the inverse of the distance from the point O to the intersection of the local tangents to the old and new crack fronts.

Let us now turn to the question of the asymptotic expression of the stresses near the original crack front (prior to propagation), the crack lips being assumed to be traction-free; in fact only the first three terms of their expansion in powers of the distance to $\mathscr{F}$ will be considered here. It is now well known that the first, singular term of this expansion can simply be obtained by superposing that corresponding to the plane strain (not plane stress!) case (mode I+mode II) and that corresponding to the antiplane case (mode III) (see e.g. Bui, 1978, for the case of a plane crack, and Leblond and Torlai, 1992 for the general case). The form of the second and third terms, however, is less well known. It was studied in detail by Leblond and Torlai (1992), who found the following expression:

$$\sigma_{ij}(x_1,x_2,x_3=0) = \frac{K_p f^p_{ij}(\theta)}{\sqrt{r}} + T_p g^p_{ij}(\theta) + [B_p h^p_{ij}(\theta) + K'_p l^p_{ij}(\theta) + C_{\lambda\mu} K_p m^{p\lambda\mu}_{ij}(\theta) + \Gamma K_p n^p_{ij}(\theta)]\sqrt{r} + O(r) \quad (i,j=1,2,3), \qquad (2)$$

where r, θ denote polar coordinates in the Ox_1x_2 plane and the Einstein summation convention is employed for the indices $p = \mathrm{I, II, III}$ and λ, $\mu = 1{,}3$. K_{I}, K_{II}, K_{III} here are the SIFs at the point O. T_{I}, T_{II}, T_{III} are the *non-singular stresses* at the same point; these quantities correspond to uniform stress fields of the form $\sigma_{11} = T_{\mathrm{I}}$, $\sigma_{13} = T_{\mathrm{II}}$ and $\sigma_{33} = T_{\mathrm{III}}$, respectively. B_{I}, B_{II}, B_{III} are coefficients which bear no special name. K'_{I}, K'_{II}, $K'_{\mathrm{III}} \equiv \mathrm{d}K_{\mathrm{I}}/\mathrm{d}s$, $\mathrm{d}K_{\mathrm{II}}/\mathrm{d}s$, $\mathrm{d}K_{\mathrm{III}}/\mathrm{d}s$ are the derivatives of the SIFs along $\mathscr{F}$ at the point O. Finally the functions f^p_{ij}, g^p_{ij}, h^p_{ij}, l^p_{ij}, $m^{p\lambda\mu}_{ij}$, n^p_{ij} are universal combinations of sines and cosines depending only on Poisson's ratio. The terms $K'_p l^p_{ij}(\theta)\sqrt{r}$, $C_{\lambda\mu}K_p m^{p\lambda\mu}_{ij}(\theta)\sqrt{r}$, $\Gamma K_p n^p_{ij}(\theta)\sqrt{r}$ are corrections arising from the non-uniformity of the SIFs along the crack front, plus the curvatures of the surface and front of the crack. Equation (2) is given for points lying in the Ox_1x_2 plane, but similar expressions hold in all planes perpendicular to the crack front, since the choice of the point O on that front is arbitrary.

The stress expansion after kinking and propagation of the crack is of the same type, provided of course that the crack front is shifted to its new position and all coordinates and geometric parameters changed accordingly.

3. Continuity of the mechanical fields with respect to the crack extension length

We now consider a finite, three-dimensional body Ω containing a traction-free crack of the type described above. (In fact the hypothesis of finiteness of Ω is made only for simplicity; it can easily be verified that all reasonings and results expounded below remain valid provided only that the *crack front* is finite.) The complementary portions $\partial\Omega_u$ and $\partial\Omega_t$ of the boundary of this body are subjected to some prescribed displacements $\mathbf{u}^p$ and tractions $\mathbf{t}^p$ which are assumed for simplicity not to vary as the crack propagates. (In fact, this hypothesis is incompatible with the demand that the criterion be satisfied at every stage of the propagation, and will be removed *in fine*.)

It will be assumed from now on that $\delta(s')$ is of the form $\varepsilon\eta(s')$ where ε is a small positive parameter and $\eta(s')$ a given, *fixed* non-negative function. The parameter ε can be thought of as some kinematic time and the function $\eta(s')$ as the corresponding rate of propagation of the crack front. (The point of view that $\eta(s')$ is a given function is purely formal, since in reality that function is unknown *a priori*.) Again, the value $\eta(s)$ of $\eta(s')$ at the point O will simply be noted η. The subject of our investigation will be the expansions of the mechanical fields (displacements and stresses) and of the SIFs in powers of ε.[1]

The aim of this section is to show that *the displacement and stresses at any given, fixed point of the body are continuous functions of* ε *at* $\varepsilon = 0$. (For simplicity, this property has already been, and will often be, termed "continuity with respect to the crack extension length δ", without any mention of the point s' where $\delta(s')$ has to be taken; the analogous expression "differentiability with respect to δ" should similarly be understood as differentiability with respect to ε.) This result should not be regarded as a mere triviality because the possible existence of some kink angle $\varphi(s')$ along the crack front means that the propagation path is geometrically singular at $\varepsilon = 0$.

We consider two situations. In the first one the loading $(\mathbf{u}^p, \mathbf{t}^p)$ is exerted on $\partial\Omega_u$ and $\partial\Omega_t$ and the (traction-free) crack is in its original position, prior to kinking and propagation. Using a classical LEFM trick, we can assume the crack to be in fact in its final position provided that suitable tractions $\mathbf{t}^{\pm}(s', x)$ are exerted on the upper $(+)$ and lower $(-)$ lips of the crack extension; the symbol x here denotes the curvilinear distance from the point s' along the intersection of the crack extension and the plane orthogonal to the original front at s'. These tractions are connected to the original SIFs and are $O(x^{-1/2})$. The displacement at any point M in this situation is denoted $\mathbf{u}(M)$. In the second situation, $\mathbf{u}^p$ and $\mathbf{t}^p$ are still imposed on $\partial\Omega_u$ and $\partial\Omega_t$ but the crack is in its final position; equivalently, the tractions $\mathbf{t}^{\pm}(s', x)$ are released. The displacement at M is then denoted $\mathbf{u}(M, \varepsilon)$.

Considering the difference between the two situations, one obtains a Problem A where a zero displacement is imposed on $\partial\Omega_u$ and a zero traction on $\partial\Omega_t$ plus the main part of the crack, while the crack extension is subjected to the tractions $-\mathbf{t}^{\pm}(s', x)$. The displacement at M is then $\mathbf{u}(M, \varepsilon) - \mathbf{u}(M)$.

We further define a Problem B in the following way: $(\mathcal{O}X_1X_2X_3)$ denoting an arbitrary, fixed orthonormal frame, a zero displacement is again imposed on $\partial\Omega_u$ as well as a zero traction on $\partial\Omega_t$ and the entire crack (main part plus extension), whereas a unit point force is exerted on the point M in the direction $\mathbf{E}_i \equiv \partial\mathcal{O}\mathbf{M}/\partial X_i$. The resulting displacements at the points (s', x) of the upper and lower lips of the crack extension are denoted $\mathbf{v}^{(i)\pm}(s', x, \varepsilon, M)$. (These displacements obviously depend on the length of the crack extension, which is why the argument ε is introduced in the notation; also, note that M does *not* represent here the point of observation of the displacement, but that of application of the force.)

[1] The aim of the introduction of the multiplicative decomposition of $\delta(s')$ in the form $\varepsilon\eta(s')$ is precisely to allow for expansions in terms of a single parameter ε instead of the function $\delta(s')$, which can be characterized only by an infinite number of parameters.

Applying Betti's reciprocity theorem to Problems A and B, one gets

$$u_i(M,\varepsilon)-u_i(M) = \int_{\mathscr{F}} \mathrm{d}s' \int_0^{\delta(s')} [-\mathbf{t}^+(s',x)\cdot\mathbf{v}^{(i)+}(s',x,\varepsilon,M) - \mathbf{t}^-(s',x)\cdot\mathbf{v}^{(i)-}(s',x,\varepsilon,M)][1+O(\varepsilon)]\,\mathrm{d}x.$$

The multiplicative term $1+O(\varepsilon)$ here arises from the expression of the elementary area on the crack extension, namely $[1+O(\varepsilon)]\,\mathrm{d}s'\,\mathrm{d}x$. Differentiating this equation with respect to the coordinates X_j of M also yields, since the tractions $\mathbf{t}^\pm$ are independent of the position of that point:

$$\frac{\partial u_i}{\partial X_j}(M,\varepsilon)-\frac{\partial u_i}{\partial X_j}(M) = \int_{\mathscr{F}} \mathrm{d}s' \int_0^{\delta(s')} \left[-\mathbf{t}^+(s',x)\cdot\frac{\partial \mathbf{v}^{(i)+}}{\partial X_j}(s',x,\varepsilon,M) - \mathbf{t}^-(s',x)\cdot\frac{\partial \mathbf{v}^{(i)-}}{\partial X_j}(s',x,\varepsilon,M)\right][1+O(\varepsilon)]\,\mathrm{d}x.$$

The quantities $\partial\mathbf{v}^{(i)\pm}/\partial X_j$ which appear in this expression do *not* represent deformations, since the coordinates with respect to which one differentiates are those of the point of application of the force, not of that of observation of the displacement; they must instead be interpreted as displacements at the points (s',x) of the upper and lower lips of the crack extension resulting from the application of unit "dipoles" at M, just as in the two-dimensional case (see Leblond, 1989).

Since the quantities $\mathbf{v}^{(i)\pm}$ and $\partial\mathbf{v}^{(i)\pm}/\partial X_j$ are displacements, they are bounded. It follows that $|u_i(M,\varepsilon)-u_i(M)|$ and $|(\partial u_i/\partial X_j)(M,\varepsilon)-(\partial u_i/\partial X_j)(M)|$ are smaller than some constants times the integral

$$\int_{\mathscr{F}} \mathrm{d}s' \int_0^{\varepsilon\eta(s')} [\|\mathbf{t}^+(s',x)\|+\|\mathbf{t}^-(s',x)\|][1+O(\varepsilon)]\,\mathrm{d}x.$$

Since the tractions here are $O(x^{-1/2})$ and $\mathscr{F}$ is bounded (this results from the assumed finiteness of Ω), this integral is $O(\sqrt{\varepsilon})$. It follows that $u_i(M,\varepsilon)-u_i(M)$ and $(\partial u_i/\partial X_j)(M,\varepsilon)-(\partial u_i/\partial X_j)(M)$ are also $O(\sqrt{\varepsilon})$, and therefore that the displacement and its gradient (and hence the stresses) at the given point M are continuous with respect to ε at $\varepsilon=0$, q.e.d.

4. First term of the expansion of the SIFs in powers of the crack extension length

We now turn to the expression of the SIFs just after the kink. We first assume that Ω is a sphere of centre O and radius R, subjected to a given surface traction field $\mathscr{T}$ on its boundary and containing a traction-free, kinked and curved *surface* crack whose original front $\mathscr{F}$ passes through the point O (Fig. 4). Let $K_{\mathrm{I}}(\varepsilon)$, $K_{\mathrm{II}}(\varepsilon)$, $K_{\mathrm{III}}(\varepsilon)$ denote the SIFs at that point O^* of the final crack front which lies in the plane orthogonal to $\mathscr{F}$ at O. These SIFs depend on all the geometric and mechanical parameters of the problem; this can be written symbolically

$$\mathbf{K}(\varepsilon) \equiv (K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}})(\varepsilon) \equiv \mathscr{L}[R,\mathbf{C},\Gamma,\varphi,\varphi',a^*,C^*,\varepsilon\eta,\eta'/\eta;\mathscr{T}] \tag{3}$$

where the functional $\mathscr{L}$ is linear with respect to the traction field $\mathscr{T}$. Also, the natural assumption

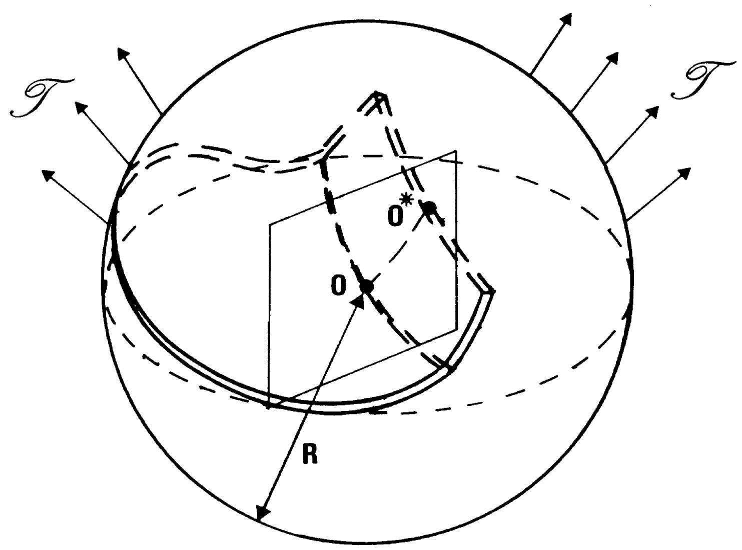

Fig. 4. Kinked and curved surface crack in a sphere.

is made that it is indefinitely differentiable with respect to all its geometric arguments, except the extension length $\varepsilon\eta$ for the value $\varepsilon\eta = 0$ (see Section 4 below). Parameters characterizing the geometry of the surface and the front of the crack with a higher degree of accuracy than that considered here should also, in theory, be included among the arguments of $\mathscr{L}$. They are omitted here for the sake of simplicity of the notation; it is easy to check *a posteriori* that introducing them does not induce any modification of the results derived below. Also, note that since we are dealing with a surface crack, $\mathscr{L}$ does not depend upon any argument (other than R) measuring the dimensions of the main part of the crack.

It is well known that a new solution to the equations of linear elasticity can be obtained from an old one by multiplying all distances and displacements by a positive factor λ while keeping the stresses (and therefore the surface tractions) unchanged. If such a transformation is performed in the present case, the geometric parameters R, $\mathbf{C}$, Γ, φ, φ', a^*, C^*, $\varepsilon\eta$, η'/η become λR, $\mathbf{C}/\lambda$, Γ/λ, φ, φ'/λ, $a^*/\sqrt{\lambda}$, C^*/λ, $\lambda\varepsilon\eta$, $\eta'/(\lambda\eta)$, respectively, and the SIFs, being limits of certain stress components times the square root of the orthogonal distance to the crack front, are multiplied by $\sqrt{\lambda}$. Thus, the functional $\mathscr{L}$ verifies the following "homogeneity" property (for every positive λ):

$$\mathscr{L}[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi, \varphi'/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \lambda\varepsilon\eta, \eta'/(\lambda\eta); \mathscr{T}] = \sqrt{\lambda}\,\mathscr{L}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \varepsilon\eta, \eta'/\eta; \mathscr{T}]. \quad (4)$$

Let $\mathscr{L}^*$ denote the limit of the functional $\mathscr{L}$ for $\varepsilon \to 0$, i.e. for a vanishing crack extension length (this is the functional that gives the SIFs just after the kink). This new functional is still linear with respect to $\mathscr{T}$, and it is indefinitely differentiable with respect to *all* its geometry arguments, since it does not, by definition, depend on $\varepsilon\eta$. With this notation, eqn (4) becomes in the limit $\varepsilon \to 0$:

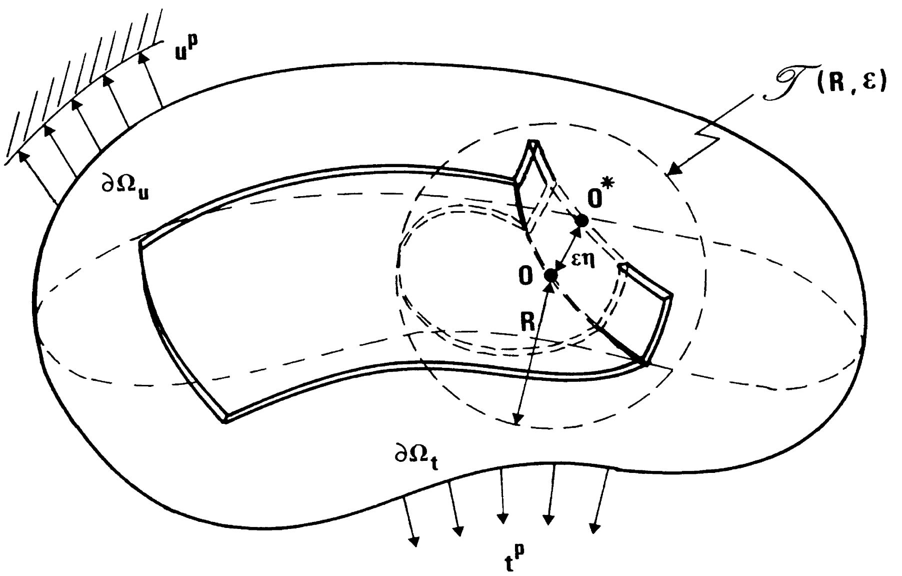

Fig. 5. Kinked and curved crack in an arbitrary body.

$$\mathscr{L}^*[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi, \varphi'/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \eta'/(\lambda\eta); \mathscr{T}] = \sqrt{\lambda}\,\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta, \eta'/\eta; \mathscr{T}]. \quad (5)$$

Thus, $\mathscr{L}^*$ verifies a homogeneity property analogous to eqn (4) for $\mathscr{L}$, but for the disappearance of the argument $\varepsilon\eta$.

We now come back to the case of an arbitrarily shaped body, and consider spheres of center O ($\in \mathscr{F}$) and sufficiently small radius R for the surface $\mathscr{S}$ of the main part of the crack to intersect their boundary (this means that the crack is a surface crack within such spheres) (Fig. 5). Let $\mathscr{T}(R, \varepsilon)$ denote the traction field exerted on the boundary of the sphere of radius R, as a result of the application of the external load $(\mathbf{u}^p, \mathbf{t}^p)$ on $\partial\Omega_u$ and $\partial\Omega_t$, when the crack extension length is $\delta(s') = \varepsilon\eta(s')$. For any such length, the mechanical fields inside the sphere, and hence the SIFs at the point O^*, remain unchanged if one eliminates its exterior while preserving the traction field $\mathscr{T}(R, \varepsilon)$ exerted on its boundary. Hence these SIFs may be expressed as

$$\mathbf{K}(\varepsilon) \equiv (K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}})(\varepsilon) = \mathscr{L}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \varepsilon\eta, \eta'/\eta; \mathscr{T}(R, \varepsilon)]. \quad (6)$$

Let us take the limit $\varepsilon \to 0$ in the preceding equation. Then $\mathscr{L}$ tends to $\mathscr{L}^*$ by definition, and $\mathscr{T}(R, \varepsilon)$ tends to the traction field $\mathscr{T}(R)$ exerted on the boundary of the sphere of radius R *prior to kinking and propagation of the crack*, because of the property of continuity of the stresses at a fixed point with respect to the crack extension length established in the preceding section. Thus one gets

$$\mathbf{K}^* \equiv \lim_{\varepsilon \to 0} \mathbf{K}(\varepsilon) = \mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}(R)]. \quad (7)$$

This equation shows that quite remarkably, the SIFs *just after* the kink depend on the loading only through the value of the stress field *prior to kinking*.

In fact, they will now be seen to depend only on the *first*, *singular term* of the stress expansion before the kink. In order to prove this, let us transform eqn (7) by using eqn (5) with $\lambda = 1/R$ plus the linearity of $\mathscr{L}^*$ with respect to $\mathscr{T}$:

$$\mathbf{K}^* = \sqrt{R}\,\mathscr{L}^*[1, R\mathbf{C}, R\Gamma, \varphi, R\varphi', \sqrt{R}\,a^*, RC^*, R\eta'/\eta; \mathscr{T}(R)]$$
$$= \mathscr{L}^*[1, R\mathbf{C}, R\Gamma, \varphi, R\varphi', \sqrt{R}\,a^*, RC^*, R\eta'/\eta; \sqrt{R}\,\mathscr{T}(R)].$$

In intuitive terms, this transformation is equivalent to "watching the sphere through a magnifying lens". Now eqn (2) implies that the surface traction $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$ exerted on the boundary of the sphere prior to kinking is of the form

$$\mathbf{t}(\rho, \psi, \chi) = K_p \mathbf{f}^p(\psi, \chi)/\sqrt{\rho} + O(1)$$

where ρ, ψ, χ denote spherical coordinates with origin at O: ρ, distance from O to the generic point M; ψ, angle between Ox_3 and the vector $\mathbf{OM}$; χ, polar angle of the projection of $\mathbf{OM}$ onto the Ox_1x_2 plane ($\equiv \theta$ if $M \in Ox_1x_2$). The dependence with respect to ψ here arises first from the expression of the orthogonal distance to $\mathscr{F}$, namely $\rho \sin\psi + O(\rho^2)$, and second from the fact that the normal vector $\mathbf{n}$ to the sphere depends on this angle. It follows that, denoting between braces the traction field defined by a certain density:

$$\sqrt{R}\,\mathscr{T}(R) \equiv K_p\{\mathbf{f}^p(\psi, \chi)\} + O(\sqrt{R}).$$

Inserting this formula into the above expression of $\mathbf{K}^*$ and taking the limit $R \to 0$ (this is licit since this expression is valid for all sufficiently small values of R), one gets

$$\mathbf{K}^* = K_p \mathscr{L}^*[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \equiv \mathbf{F}(\varphi)\cdot\mathbf{K} \tag{8}$$

where $\mathbf{F}(\varphi)$ is a linear operator depending on the kink angle. In intuitive terms, the final disappearance of all geometric parameters except φ arises from the fact that they vanish in the limit of "infinite magnification of the lens".

This expression of $\mathbf{K}^*$ will be termed *universal* in the sense that it applies to any situation: whatever the complexity of the geometry and the loading, the SIFs just after the kink depend on the various geometric and mechanical parameters of the problem *only through the local SIFs just before the kink plus the local kink angle*. This result generalizes that previously established in the two-dimensional case (plane strain conditions) (Leblond, 1989; Leguillon, 1993).

In order to calculate the components $F_{\mathrm{I,I}}(\varphi)$, $F_{\mathrm{I,II}}(\varphi)$, $F_{\mathrm{II,I}}(\varphi)$, $F_{\mathrm{II,II}}(\varphi)$ of the operator $\mathbf{F}(\varphi)$ (the commas here only serve to separate the indices and do not denote a differentiation), one may take advantage of their universality property by considering the simplest possible particular case, namely that of an infinite body under plane strain conditions, loaded by uniform tractions at infinity and containing a straight crack extended in an arbitrary direction by an infinitesimal straight extension. This problem was studied in several papers, notably Bilby and Cardew (1975), Wu (1978), Amestoy et al. (1979) and Amestoy and Leblond (1992). None of these yielded completely explicit formulae for the functions $F_{\mathrm{I,I}}(\varphi)$, $F_{\mathrm{I,II}}(\varphi)$, $F_{\mathrm{II,I}}(\varphi)$, $F_{\mathrm{II,II}}(\varphi)$; the work which goes farthest in the search for analytic expressions of these functions is the last one, which provides induction formulae that allow for the derivation of their exact expansions up to an arbitrary order with respect to the kink angle. These expansions are given up to order 20 in the Appendix [eqns (A2)–(A5)].

The function $F_{\mathrm{III,III}}(\varphi)$ can be obtained in a similar way by studying the same particular case, but with an antiplane loading. This was done as early as 1965 by Sih, for a crack extension of arbitrary length. The calculation was much simpler than in the plane strain case and a completely explicit formula for $K_{\mathrm{III}}(\varepsilon)$ could be obtained. In the limit of an infinitesimal extension, this formula yields eqn (A6) of the Appendix.

Finally, a simple symmetry argument shows that the remaining functions $F_{\mathrm{I,III}}(\varphi)$, $F_{\mathrm{II,III}}(\varphi)$, $F_{\mathrm{III,I}}(\varphi)$, $F_{\mathrm{III,II}}(\varphi)$ are all zero, i.e. that the operator $\mathbf{F}(\varphi)$ is of the following form:

$$\mathbf{F}(\varphi) \equiv \begin{bmatrix} F_{\mathrm{I,I}}(\varphi) & F_{\mathrm{I,II}}(\varphi) & 0 \\ F_{\mathrm{II,I}}(\varphi) & F_{\mathrm{II,II}}(\varphi) & 0 \\ 0 & 0 & F_{\mathrm{III,III}}(\varphi) \end{bmatrix}. \tag{9}$$

Indeed, let us apply a symmetry with respect to the Ox_1x_2 plane to both the body and the mechanical fields (Fig. 6). For points lying in that plane, the u_1 and u_2 components of the displacement are left unmodified while the u_3 component changes sign. It follows that K_{I}, K_{II}, K_{III}, K_{I}^*, K_{II}^*, K_{III}^* become K_{I}, K_{II}, $-K_{\mathrm{III}}$, K_{I}^*, K_{II}^*, $-K_{\mathrm{III}}^*$, respectively. Since φ is unchanged, the formulae connecting the K_p^* to the K_p read

$$K_{\mathrm{I}}^* = F_{\mathrm{I,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}} + F_{\mathrm{I,III}}(\varphi)K_{\mathrm{III}};$$

$$K_{\mathrm{II}}^* = F_{\mathrm{II,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} + F_{\mathrm{II,III}}(\varphi)K_{\mathrm{III}};$$

$$K_{\mathrm{III}}^* = F_{\mathrm{III,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{III,II}}(\varphi)K_{\mathrm{II}} + F_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}}$$

before the transformation and

$$K_{\mathrm{I}}^* = F_{\mathrm{I,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}} - F_{\mathrm{I,III}}(\varphi)K_{\mathrm{III}};$$

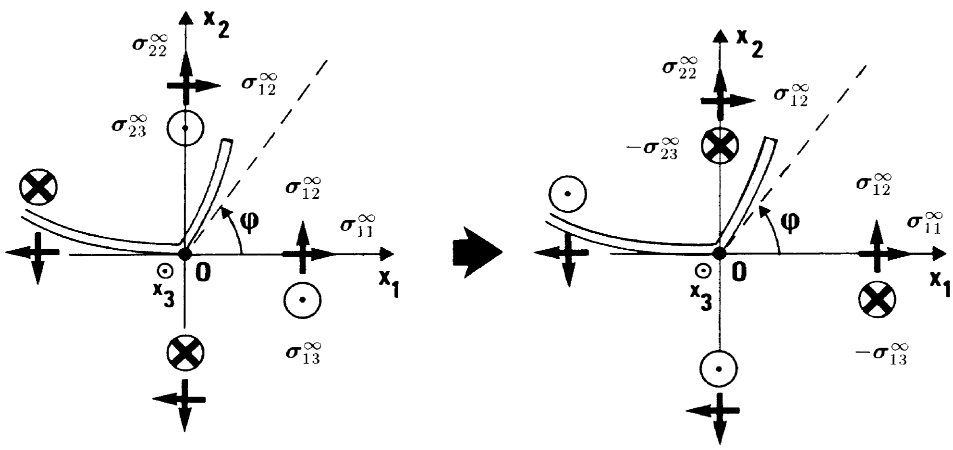

Fig. 6. Effect of a symmetry about the Ox_1x_2 plane on the geometry and the loading.

$$K_{\mathrm{II}}^* = F_{\mathrm{II,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} - F_{\mathrm{II,III}}(\varphi)K_{\mathrm{III}};$$
$$-K_{\mathrm{III}}^* = F_{\mathrm{III,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{III,II}}(\varphi)K_{\mathrm{II}} - F_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}}$$

after it. It immediately follows that $F_{\mathrm{I,III}}(\varphi) = F_{\mathrm{II,III}}(\varphi) = F_{\mathrm{III,I}}(\varphi) = F_{\mathrm{III,II}}(\varphi) = 0$. (A similar argument involving a symmetry with respect to the Ox_1x_3 plane can be used to show that $F_{\mathrm{I,I}}(\varphi)$, $F_{\mathrm{II,II}}(\varphi)$, $F_{\mathrm{III,III}}(\varphi)$ are even, and $F_{\mathrm{I,II}}(\varphi)$, $F_{\mathrm{II,I}}(\varphi)$ odd, functions of φ.)

5. Differentiability of the mechanical fields with respect to the crack extension length

We now wish to show that with the notations of Section 3, $\mathbf{u}(M,\varepsilon)$ *and* $\boldsymbol{\sigma}(M,\varepsilon)$ *are differentiable functions of* ε *at* $\varepsilon = 0$. The proof will rely on the following elementary mathematical result:

Proposition: Let $f(x)$ denote a real function of a real variable, defined for $x \geqslant 0$, continuous at $x = 0$, differentiable for $x > 0$, and such that $f'(x)$ tends to a limit l for $x \to 0$. Then f admits a (right-hand) derivative equal to l at $x = 0$.

We shall apply this result to the quantities $u_i(M,\varepsilon)$ and $(\partial u_i/\partial X_j)(M,\varepsilon)$, considered as functions of ε. (As in Section 3, coordinates and components will refer here to a fixed, arbitrary orthonormal frame $(\mathcal{O}X_1X_2X_3)$.)

First, both $u_i(M,\varepsilon)$ and $(\partial u_i/\partial X_j)(M,\varepsilon)$ have been shown in Section 3 to be continuous with respect to ε at $\varepsilon = 0$.

Second, Rice's (1985) formulation of the theory of Bueckner's weight functions is applicable for $\varepsilon > 0$, since the crack propagates in a regular manner after the initial kink. It follows that $u_i(M,\varepsilon)$ is differentiable with respect to ε for $\varepsilon > 0$, its derivative being given by

$$\frac{\partial u_i}{\partial \varepsilon}(M,\varepsilon > 0) = \int_{\mathscr{F}^*} \left\{ \frac{2(1-\nu^2)}{E} [K_{\mathrm{I}}(s'^*,\varepsilon)K_{\mathrm{I}}^{(i)}(\Omega,s'^*,\varepsilon,M) + K_{\mathrm{II}}(s'^*,\varepsilon)K_{\mathrm{II}}^{(i)}(\Omega,s'^*,\varepsilon,M)] + \frac{2(1+\nu)}{E} K_{\mathrm{III}}(s'^*,\varepsilon)K_{\mathrm{III}}^{(i)}(\Omega,s'^*,\varepsilon,M) \right\} [1+O(\varepsilon)]\eta(s'^*)\,\mathrm{d}s'^*. \quad (10)$$

In this expression E and ν denote Young's modulus and Poisson's ratio, $\mathscr{F}^*$ the front of the extended crack, s'^* the curvilinear distance along $\mathscr{F}^*$, and $\eta(s'^*)$ is just $\eta(s')$ expressed in terms of s'^* instead of s'. Also, $K_{\mathrm{I}}(s'^*,\varepsilon)$, $K_{\mathrm{II}}(s'^*,\varepsilon)$, $K_{\mathrm{III}}(s'^*,\varepsilon)$ denote the SIFs at the point s'^* of $\mathscr{F}^*$ (for a crack extension of length $\delta(s') = \varepsilon\eta(s')$), and $K_{\mathrm{I}}^{(i)}(\Omega,s'^*,\varepsilon,M)$, $K_{\mathrm{II}}^{(i)}(\Omega,s'^*,\varepsilon,M)$, $K_{\mathrm{III}}^{(i)}(\Omega,s'^*,\varepsilon,M)$ those which would arise, at the same point, from application of a unit point force in the direction $\mathbf{E}_i \equiv \partial\mathcal{O}\mathbf{M}/\partial X_i$ at the point M, $\partial\Omega_u$ and $\partial\Omega_t$ being simultaneously subjected to zero displacements and tractions, respectively (the argument "Ω" is introduced here to underline the dependence upon the geometry of the whole body *via* these boundary conditions). The term $1+O(\varepsilon)$ arises from the fact that the line along which $\delta(s')$ is measured, namely the intersection of the crack extension and the plane orthogonal to $\mathscr{F}$ at the point s', is not exactly orthogonal to $\mathscr{F}^*$; thus, the "*normal* velocity" (which appears in Rice's formula) of the crack front at the point s'^* of $\mathscr{F}^*$ is not exactly $\eta(s'^*)$ but $[1+O(\varepsilon)]\eta(s'^*)$.

Differentiating eqn (10) with respect to the coordinates X_j of M, we also get

$$\frac{\partial}{\partial \varepsilon}\left(\frac{\partial u_i}{\partial X_j}\right)(M, \varepsilon > 0) = \int_{\mathscr{F}^*} \left\{\frac{2(1-v^2)}{E}\left[K_{\mathrm{I}}(s'^*, \varepsilon)\frac{\partial K_{\mathrm{I}}^{(i)}(\Omega, s'^*, \varepsilon, M)}{\partial X_j}\right.\right.$$
$$\left.\left.+K_{\mathrm{II}}(s'^*, \varepsilon)\frac{\partial K_{\mathrm{II}}^{(i)}(\Omega, s'^*, \varepsilon, M)}{\partial X_j}\right] + \frac{2(1+v)}{E}K_{\mathrm{III}}(s'^*, \varepsilon)\frac{\partial K_{\mathrm{III}}^{(i)}(\Omega, s'^*, \varepsilon, M)}{\partial X_j}\right\}$$
$$\times [1+O(\varepsilon)]\eta(s'^*)\,\mathrm{d}s'^*. \quad (11)$$

Third, when ε tends toward zero, the right-hand sides of eqns (10), (11) tend to the limits

$$\int_{\mathscr{F}} \left\{\frac{2(1-v^2)}{E}[K_{\mathrm{I}}^*(s')K_{\mathrm{I}}^{(i)*}(\Omega, s', M) + K_{\mathrm{II}}^*(s')K_{\mathrm{II}}^{(i)*}(\Omega, s', M)]\right.$$
$$\left.+\frac{2(1+v)}{E}K_{\mathrm{III}}^*(s')K_{\mathrm{III}}^{(i)*}(\Omega, s', M)\right\}\eta(s')\,\mathrm{d}s'$$

and

$$\int_{\mathscr{F}} \left\{\frac{2(1-v^2)}{E}\left[K_{\mathrm{I}}^*(s')\frac{\partial K_{\mathrm{I}}^{(i)*}(\Omega, s', M)}{\partial X_j} + K_{\mathrm{II}}^*(s')\frac{\partial K_{\mathrm{II}}^{(i)*}(\Omega, s', M)}{\partial X_j}\right]\right.$$
$$\left.+\frac{2(1+v)}{E}K_{\mathrm{III}}^*(s')\frac{\partial K_{\mathrm{III}}^{(i)*}(\Omega, s', M)}{\partial X_j}\right\}\eta(s')\,\mathrm{d}s'$$

where the $K_p^{(i)*}(\Omega, s', M)$ denote the limits of the $K_p^{(i)}(\Omega, s'^*, \varepsilon, M)$ for $\varepsilon \to 0$, i.e. the SIFs just after the kink resulting from the loadings described above, just as for the $K_p(s'^*, \varepsilon)$ and $K_p^*(s')$.

Using the above proposition, one concludes that both $u_i(M, \varepsilon)$ and $(\partial u_i/\partial X_j)(M, \varepsilon)$ [and hence $\sigma_{ij}(M, \varepsilon)$] are differentiable with respect to ε at $\varepsilon = 0$, their derivatives being given by

$$\frac{\partial u_i}{\partial \varepsilon}(M, \varepsilon = 0) = \int_{\mathscr{F}} \left\{\frac{2(1-v^2)}{E}[K_{\mathrm{I}}^*(s')K_{\mathrm{I}}^{(i)*}(\Omega, s', M) + K_{\mathrm{II}}^*(s')K_{\mathrm{II}}^{(i)*}(\Omega, s', M)]\right.$$
$$\left.+\frac{2(1+v)}{E}K_{\mathrm{III}}^*(s')K_{\mathrm{III}}^{(i)*}(\Omega, s', M)\right\}\eta(s')\,\mathrm{d}s'; \quad (12)$$

$$\frac{\partial}{\partial \varepsilon}\left(\frac{\partial u_i}{\partial X_j}\right)(M, \varepsilon = 0) = \int_{\mathscr{F}} \left\{\frac{2(1-v^2)}{E}\left[K_{\mathrm{I}}^*(s')\frac{\partial K_{\mathrm{I}}^{(i)*}(\Omega, s', M)}{\partial X_j}\right.\right.$$
$$\left.\left.+K_{\mathrm{II}}^*(s')\frac{\partial K_{\mathrm{II}}^{(i)*}(\Omega, s', M)}{\partial X_j}\right] + \frac{2(1+v)}{E}K_{\mathrm{III}}^*(s')\frac{\partial K_{\mathrm{III}}^{(i)*}(\Omega, s', M)}{\partial X_j}\right\}\eta(s')\,\mathrm{d}s'. \quad (13)$$

These differentiability properties imply that $\mathbf{u}(M, \varepsilon)$ and $\boldsymbol{\sigma}(M, \varepsilon)$ admit the following asymptotic expressions for $\varepsilon \to 0$:

$$\mathbf{u}(M, \varepsilon) = \mathbf{u}(M) + O(\varepsilon); \quad \boldsymbol{\sigma}(M, \varepsilon) = \boldsymbol{\sigma}(M) + O(\varepsilon), \quad (14)$$

where the notations $\mathbf{u}(M)$ and $\boldsymbol{\sigma}(M)$ refer to the situation prior to kinking. Incidentally, one may remark that the next terms in the right-hand sides here are *not proportional to* ε^2 *but to* $\varepsilon^{3/2}$; this results from integration of eqns (10), (11) using the fact evidenced below, and already well known in the two-dimensional case, that the asymptotic expression of the $K_p(s'^*, \varepsilon)$ is of the type $K_p^*(s') + (\cdots)\sqrt{\varepsilon}$ (and not $K_p^*(s') + (\cdots)\varepsilon$). Therefore *the displacement and stresses are not twice differentiable with respect to the crack extension length*. This clearly shows that again, the assertions that *they are continuous and* (*once*) *differentiable* cannot merely be accepted as "obvious" and need to be proved in a rigorous manner, as was done in Section 3 and here.

6. Second term of the expansion of the SIFs in powers of the crack extension length

The second term of the expansion of the SIFs in powers of ε will now be studied by extending the analysis of Section 4 to the next order in ε. We first introduce the asymptotic expression of the functional $\mathscr{L}$ for $\varepsilon \to 0$:

$$\mathscr{L}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \varepsilon\eta, \eta'/\eta; \mathscr{T}] = \mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}] + \mathscr{L}^{(1/2)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}]\sqrt{\varepsilon\eta} + O(\varepsilon). \quad (15)$$

The fact that the second term of the expression of $\mathscr{L}$ is proportional to $\sqrt{\varepsilon}$ (instead of ε as one would *a priori* expect) is due to the existence of the kink characterized by the angle φ, and can be justified in the same way as in the two-dimensional case (see Leblond, 1989). The arguments φ', a^*, C^*, η'/η in the expression of $\mathscr{L}^*$ have now been discarded, since formula $(8)_2$ shows that $\mathbf{K}^*$, and hence $\mathscr{L}^*$, depend upon the geometric parameters of the crack extension only through the kink angle φ.[2] Just as $\mathscr{L}^*$, the function $\mathscr{L}^{(1/2)}$ is linear in $\mathscr{T}$ and indefinitely differentiable with respect to all its geometric arguments.

Expanding eqn (4) in powers of ε, one gets

$$\mathscr{L}^*[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi; \mathscr{T}] + \mathscr{L}^{(1/2)}[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi, \varphi'/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \eta'/(\lambda\eta); \mathscr{T}]\sqrt{\lambda\varepsilon\eta} + O(\varepsilon)$$
$$= \sqrt{\lambda}\{\mathscr{L}^*[R, \mathbf{C}, \Gamma, \varphi; \mathscr{T}] + \mathscr{L}^{(1/2)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}]\sqrt{\varepsilon\eta} + O(\varepsilon)\};$$

identification of the terms proportional to $\sqrt{\varepsilon}$ then yields

$$\mathscr{L}^{(1/2)}[\lambda R, \mathbf{C}/\lambda, \Gamma/\lambda, \varphi, \varphi'/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \eta'/(\lambda\eta); \mathscr{T}] = \mathscr{L}^{(1/2)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}]. \quad (16)$$

Thus, the homogeneity property for $\mathscr{L}^{(1/2)}$ differs from those for $\mathscr{L}$ and $\mathscr{L}^*$ [eqns (4), (5)] by a factor of $\sqrt{\lambda}$.

Equation $(14)_2$ implies that the traction field $\mathscr{T}(R, \varepsilon)$ exerted on the boundary of the sphere of

[2] Note that $\mathscr{L}^*$ still depends on $\mathbf{C}$ and Γ, however, because the SIFs just after the kink depend on those just before it which, when expressed as functions of the loading $\mathscr{T}$, depend on the curvatures of the surface and the front of the initial crack.

radius R as a result of the external loading, when the length of the crack extension is $\delta(s') = \varepsilon\eta(s')$, admits an asymptotic expression for $\varepsilon \to 0$ of the type

$$\mathscr{T}(R, \varepsilon) = \mathscr{T}(R) + O(\varepsilon).$$

Inserting this formula and eqn (15) into the expression (6) of $\mathbf{K}(\varepsilon)$, one gets

$$\mathbf{K}(\varepsilon) = \mathbf{K}^* + \mathbf{K}^{(1/2)}\sqrt{\varepsilon\eta} + O(\varepsilon) = \mathbf{K}^* + \mathbf{K}^{(1/2)}\sqrt{\delta} + O(\varepsilon), \tag{17}$$

where $\mathbf{K}^*$ is given by eqn (7) and $\mathbf{K}^{(1/2)}$ by

$$\mathbf{K}^{(1/2)} = \mathscr{L}^{(1/2)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}(R)]. \tag{18}$$

It thus appears that $\mathbf{K}^{(1/2)}$, just like $\mathbf{K}^*$, depends on the loading only through the value of the stress field *prior to kinking*. This is a direct consequence of the absence of a term proportional to $\sqrt{\varepsilon}$ in the asymptotic expression of $\mathscr{T}(R, \varepsilon)$, i.e. of the differentiability of the stresses at a fixed point with respect to the crack extension length. It is also remarkable that the function $\delta(s')$ enters the two-term expansion (17) of $\mathbf{K}(\varepsilon)$ *only through its value at the point O*, i.e. at $s' = s$. As already mentioned in the Introduction, this property will not subsist for higher order terms of this expansion. It will be explained in detail in Part II, but it can already be anticipated, that this is because such terms involve the derivative of $\mathscr{T}(R, \varepsilon)$ with respect to ε at $\varepsilon = 0$, which was seen in the preceding section to depend upon values of the function $\eta(s')$ *at all points of the crack front*, not only at $s' = s$ [see eqn (13)].

Let us now let R tend to zero. We first use eqn (16) with $\lambda = 1/R$ and expand the functional $\mathscr{L}$ in powers of R:

$$\mathscr{L}^{(1/2)}[R, \mathbf{C}, \Gamma, \varphi, \varphi', a^*, C^*, \eta'/\eta; \mathscr{T}] = \mathscr{L}^{(1/2)}[1, R\mathbf{C}, R\Gamma, \varphi, R\varphi', \sqrt{R}\,a^*, RC^*, R\eta'/\eta; \mathscr{T}]$$

$$= \mathscr{L}^{(1/2)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \mathscr{T}] + \sqrt{R}\,a^* \frac{\partial \mathscr{L}^{(1/2)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \mathscr{T}] + O(R).$$

Second we note that eqn (2) implies that $\mathscr{T}(R)$ admits an asymptotic expression for $R \to 0$ of the type

$$\mathscr{T}(R) \equiv \frac{K_p}{\sqrt{R}}\{\mathbf{f}^p(\psi, \chi)\} + T_p\{\mathbf{g}^p(\psi, \chi)\} + O(\sqrt{R})$$

with the same notations as in Section 4 (recall that the T_p denote the initial non-singular stresses). We then insert these expressions into eqn (18) and get

$$\mathbf{K}^{(1/2)} = \frac{1}{\sqrt{R}} \cdot K_p \mathscr{L}^{(1/2)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}]$$

$$+ T_p \mathscr{L}^{(1/2)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{g}^p(\psi, \chi)\}]$$

$$+ a^* K_p \frac{\partial \mathscr{L}^{(1/2)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] + O(\sqrt{R}).$$

Now $\mathbf{K}^{(1/2)}$ is independent of R; hence the divergent term proportional to $R^{-1/2}$ in the above expression must necessarily be zero. Taking the limit $R \to 0$, one then gets

$$\begin{aligned}\mathbf{K}^{(1/2)} &= T_p \mathscr{L}^{(1/2)}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{g}^p(\psi, \chi)\}] \\ &\quad + a^* K_p \frac{\partial \mathscr{L}^{(1/2)}}{\partial a^*}[1, \mathbf{0}, 0, \varphi, 0, 0, 0, 0; \{\mathbf{f}^p(\psi, \chi)\}] \\ &\equiv \mathbf{G}(\varphi) \cdot \mathbf{T} + a^* \mathbf{H}(\varphi) \cdot \mathbf{K} \end{aligned} \quad (19)$$

where $\mathbf{G}(\varphi)$ and $\mathbf{H}(\varphi)$ are linear operators depending on the kink angle and $\mathbf{T} \equiv (T_{\mathrm{I}}, T_{\mathrm{II}}, T_{\mathrm{III}})$ is the vector of initial non-singular stresses. Thus, $\mathbf{K}^{(1/2)}$ admits a *universal* expression analogous to, though somewhat more complex than, that of $\mathbf{K}^*$ (eqn (8)). Again, this result generalizes that already known in the two-dimensional case (Leblond, 1989; Leguillon, 1993).

Proceeding in the same way as for the operator $\mathbf{F}(\varphi)$, one can identify some components of $\mathbf{G}(\varphi)$ and $\mathbf{H}(\varphi)$ by studying the special problem of an infinite two-dimensional body loaded by uniform far tractions and containing a two-branch crack. Application of a plane loading (plane strain conditions) results in non-zero values for the first and second SIFs and for the first non-singular stress (corresponding to a uniform stress field $\sigma_{11} = T_{\mathrm{I}}$) and thus allows for the calculation of $G_{\mathrm{I,I}}(\varphi)$, $G_{\mathrm{II,I}}(\varphi)$, $H_{\mathrm{I,I}}(\varphi)$, $H_{\mathrm{I,II}}(\varphi)$, $H_{\mathrm{II,I}}(\varphi)$ and $H_{\mathrm{II,II}}(\varphi)$. Similarly, application of an antiplane loading results in non-zero values for the third SIF and the second non-singular stress (representing a uniform σ_{13} stress), and, therefore, leads to the evaluation of $G_{\mathrm{III,II}}(\varphi)$ and $H_{\mathrm{III,III}}(\varphi)$. Consideration of a straight ($a^* = 0$) secondary branch is sufficient to obtain the components of $\mathbf{G}(\varphi)$, but introduction of a non-zero a^* is necessary to get those of $\mathbf{H}(\varphi)$, since this operator describes the effect of that curvature parameter upon $\mathbf{K}^{(1/2)}$. The resulting curved crack problem can be dealt with through a first order perturbative procedure with respect to a^*, using the fact that only the first power of this parameter appears in the universal expression (19).

The case of a plane loading was treated in detail by Amestoy and Leblond (1992) for both straight and curved extensions. Again, the treatment did not yield completely explicit expressions, but only accurate high order expansions and/or numerical values, of the functions looked for. These expansions and numerical values are given in the Appendix [eqns (A7), (A8) and Table A1]. The case of an antiplane loading and a *straight* extension was studied by Sih (1965), who found eqn (A9) of the Appendix. Finally, the case of an antiplane loading and a *curved* extension was considered by the author in an unpublished work, using the same perturbative method as Amestoy and Leblond (1992); the result for the function $H_{\mathrm{III,III}}(\varphi)$ is eqn (A10) of the Appendix.

The remaining functions $G_{\mathrm{I,II}}(\varphi)$, $G_{\mathrm{I,III}}(\varphi)$, $G_{\mathrm{II,II}}(\varphi)$, $G_{\mathrm{II,III}}(\varphi)$, $G_{\mathrm{III,I}}(\varphi)$, $G_{\mathrm{III,III}}(\varphi)$, $H_{\mathrm{I,III}}(\varphi)$, $H_{\mathrm{II,III}}(\varphi)$, $H_{\mathrm{III,I}}(\varphi)$, $H_{\mathrm{III,II}}(\varphi)$ are all zero, which means that the operators $\mathbf{G}(\varphi)$ and $\mathbf{H}(\varphi)$ are of the form

$$\mathbf{G}(\varphi) \equiv \begin{bmatrix} G_{\mathrm{I,I}}(\varphi) & 0 & 0 \\ G_{\mathrm{II,I}}(\varphi) & 0 & 0 \\ 0 & G_{\mathrm{III,II}}(\varphi) & 0 \end{bmatrix}; \quad \mathbf{H}(\varphi) \equiv \begin{bmatrix} H_{\mathrm{I,I}}(\varphi) & H_{\mathrm{I,II}}(\varphi) & 0 \\ H_{\mathrm{II,I}}(\varphi) & H_{\mathrm{II,II}}(\varphi) & 0 \\ 0 & 0 & H_{\mathrm{III,III}}(\varphi) \end{bmatrix}. \quad (20)$$

For $G_{\mathrm{I,II}}(\varphi)$, $G_{\mathrm{II,II}}(\varphi)$, $G_{\mathrm{III,I}}(\varphi)$, $H_{\mathrm{I,III}}(\varphi)$, $H_{\mathrm{II,III}}(\varphi)$, $H_{\mathrm{III,I}}(\varphi)$ and $H_{\mathrm{III,II}}(\varphi)$, this results from a simple symmetry argument analogous to that presented in Section 4 for some components of the operator $\mathbf{F}(\varphi)$. For $G_{\mathrm{I,III}}(\varphi)$, $G_{\mathrm{II,III}}(\varphi)$ and $G_{\mathrm{III,III}}(\varphi)$, it follows from consideration of the case where the geometry is invariant in the direction of the (straight) crack front. Indeed one can then add a uniform σ_{33} stress to the stress field and obtain a new solution of the equations of elasticity still respecting the conditions of zero traction on the crack lips; in this process T_{III} (which precisely represents a uniform σ_{33} stress) is changed whereas neither the initial nor the final SIFs are. It immediately follows that $G_{\mathrm{I,III}}(\varphi) = G_{\mathrm{II,III}}(\varphi) = G_{\mathrm{III,III}}(\varphi) = 0$.

7. Extension of the cotterell–rice stability analysis to three-dimensional and internally loaded cracks

The preceding results will now be applied to the problem of expressing a criterion for stable coplanar propagation of a plane crack loaded in mode I by an internal pressure (in addition to the loading imposed on the external boundary of the body), generalizing that given by Cotterell and Rice (1980) for the case of plane strain conditions and a traction-free crack.

This requires to extend eqns (8), (17), (19) to the case where a pressure p is exerted on the crack lips. This is easily done by using a classical superposition argument. Indeed, let us decompose the original elasticity problem involving the external loading $(\mathbf{u}^p, \mathbf{t}^p)$ on $\partial\Omega_u$ and $\partial\Omega_t$ plus the internal pressure p into two problems (noted A and B) in the following way:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B, \quad \boldsymbol{\sigma}^A = \boldsymbol{\sigma} + p\mathbf{1}, \quad \boldsymbol{\sigma}^B = -p\mathbf{1}.$$

The stress field $\boldsymbol{\sigma}^B$ being uniform, the corresponding SIFs are zero. Therefore, with obvious notations,

$$K_p = K_p^A + K_p^B = K_p^A; \quad K_p(\varepsilon) = K_p^A(\varepsilon) + K_p^B(\varepsilon) = K_p^A(\varepsilon) \quad (p = \mathrm{I}, \mathrm{II}, \mathrm{III}).$$

It is also evident from the definition of the non-singular stresses (see Section 2) that

$$T_{\mathrm{I}}^B = T_{\mathrm{III}}^B = -p; \quad T_{\mathrm{II}}^B = 0.$$

It follows that

$$T_{\mathrm{I}} = T_{\mathrm{I}}^A - p; \quad T_{\mathrm{II}} = T_{\mathrm{II}}^A; \quad T_{\mathrm{III}} = T_{\mathrm{III}}^A - p.$$

Now the stress field $\boldsymbol{\sigma}^A$ corresponds to a problem where no traction is imposed on the crack lips. Therefore eqns (8), (17), (19) are applicable to this problem[3] and yield

$$\begin{aligned} K_p(\varepsilon) &= K_p^A(\varepsilon) = F_{pq}(\varphi)K_q^A + [G_{p,\mathrm{I}}(\varphi)T_{\mathrm{I}}^A + a^* H_{pq}(\varphi)K_q^A]\sqrt{\delta} + O(\varepsilon) \\ &= F_{pq}(\varphi)K_q + [G_{p,\mathrm{I}}(\varphi)(T_{\mathrm{I}} + p) + a^* H_{pq}(\varphi)K_q]\sqrt{\delta} + O(\varepsilon) \quad (p, q = \mathrm{I}, \mathrm{II}); \\ K_{\mathrm{III}}(\varepsilon) &= K_{\mathrm{III}}^A(\varepsilon) = F_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}}^A + [G_{\mathrm{III,II}}(\varphi)T_{\mathrm{II}}^A + a^* H_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}}^A]\sqrt{\delta} + O(\varepsilon) \\ &= F_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}} + [G_{\mathrm{III,II}}(\varphi)T_{\mathrm{II}} + a^* H_{\mathrm{III,III}}(\varphi)K_{\mathrm{III}}]\sqrt{\delta} + O(\varepsilon), \end{aligned} \tag{21}$$

[3] More precisely, recall that these equations hold provided also that the external loading is kept constant while the crack propagates. The loading of Problem A consists of the traction $\mathbf{t}^p + p\mathbf{n}$ exerted on $\partial\Omega_t$, which is indeed constant, plus the traction $\boldsymbol{\sigma}\cdot\mathbf{n} + p\mathbf{n}$ exerted on $\partial\Omega_u$. The latter traction unfortunately depends on ε since $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(M, \varepsilon)$ depends on it; however it varies like ε, not $\sqrt{\varepsilon}$, as shown in Section 5. Hence that variation of the SIFs due to the variation of the loading (which adds to that arising from propagation of the crack) is also $O(\varepsilon)$ and therefore negligible here, since we consider only terms of order $\varepsilon^0 = 1$ and $\varepsilon^{1/2}$ in the expansion of the SIFs.

where account has been taken of eqns (9) and (20). Thus, the expression of $K_{III}(\varepsilon)$ is unchanged with respect to the case of a traction-free crack, whereas those of $K_I(\varepsilon)$ and $K_{II}(\varepsilon)$ are modified through the additional terms $G_{I,I}(\varphi)p\sqrt{\delta}$ and $G_{II,I}(\varphi)p\sqrt{\delta}$.

With this sole modification, Cotterell and Rice's (1980) two-dimensional reasoning is easily transposed to three-dimensional situations; let us briefly recall that reasoning for completeness. Let us consider the case where the crack is plane and loaded in mode I, and assume that a small imperfection (slight modification of the loading or tiny deviation of the crack from coplanarity due to a local toughness inhomogeneity for instance) generates a bit of mode II. This induces a small kink. Propagation being supposed to obey Goldstein and Salganik's (1974) "principle of local symmetry" which stipulates that $K_{II}(\varepsilon)$ must be identically zero as the crack extends, one must equate the successive terms of the expansion of that quantity [eqn $(21)_1$ with $p = \text{II}$] in powers of ε or δ to zero. The first condition, $F_{II,q}(\varphi)K_q = 0$, yields the value of the (small) kink angle φ, which is in fact not needed here. The second condition gives the value of the first curvature parameter a^* of the crack extension:

$$G_{II,I}(\varphi)(T_I+p)+a^*H_{II,q}(\varphi)K_q = 0 \Rightarrow a^* = -\frac{G_{II,I}(\varphi)(T_I+p)}{H_{II,q}(\varphi)K_q},$$

i.e., since $G_{II,I}(\varphi) = -2\sqrt{2/\pi}\,\varphi+O(\varphi^3)$, $H_{II,I}(\varphi) = 3/4+O(\varphi^2)$ and $H_{II,II}(\varphi) = O(\varphi)$ (Cotterell and Rice, 1980; Amestoy and Leblond, 1992):

$$a^* = \frac{8}{3}\sqrt{\frac{2}{\pi}}\,\frac{T_I+p}{K_I}\varphi$$

to the lowest order in φ. Now coplanar propagation can be considered as stable in a first approximation if the effect of a^* tends to counterbalance that of φ, i.e. if these quantities are of opposite signs; this leads to the following stability criterion:

$$T_I+p < 0, \tag{22}$$

which is the same as that given by Cotterell and Rice (1980) but for the additional pressure term.

The above form of the stability criterion is *universal* in the sense that it is applicable to any three-dimensional situation (for a plane crack loaded in mode I), the coefficients before T_I and p being always precisely equal to unity. It is not, however, the most convenient form in practice; indeed it does not clearly exhibit the influence of the pressure applied, because the non-singular stress T_I implicitly depends on that parameter. In order to explicitly display this influence, let us decompose T_I in the following way:

$$T_I = T_I^0-\kappa p, \tag{23}$$

where T_I^0 denotes the non-singular stress due to the sole external loading $(\mathbf{u}^p, \mathbf{t}^p)$ exerted on $\partial\Omega_u$ and $\partial\Omega_t$ (the crack lips being traction-free), and $-\kappa p$ that due to the internal pressure alone (zero displacements and tractions being imposed on $\partial\Omega_u$ and $\partial\Omega_t$). The $-$ sign is introduced here because this non-singular stress will appear in practice to be negative. The stability criterion then takes the form

$$T_{\mathrm{I}}^{0}+(1-\kappa)p<0. \tag{24}$$

The κ coefficient here has no universal expression and must be calculated in each particular case.

8. Application to hydraulic fracturing

We now wish to apply the preceding result to the study of crack path stability vs out-of-plane deviations in hydraulic fracturing. The crack is idealized as penny-shaped and the surrounding medium as infinite. The crack is subjected to an internal pressure p and some uniform (negative) stresses are applied at infinity. We restrict our investigation to the case where the initial SIF K_{II} is zero all along the crack front; the physical justification of this hypothesis is that if it is not, immediate kinking of the crack will occur as soon as propagation starts and tend to reduce it down to zero. This assumption imposes that the normal Oz to the crack plane be parallel to a principal direction of the stress tensor at infinity. We then choose, within the crack plane, some axes Ox and Oy parallel to the other two principal directions; we shall also use the associated cylindrical coordinates r, θ, z. The situation is summarized in Fig. 7, where the principal stresses at infinity are denoted $-P$, $-Q$, $-R$ (P, Q and R being positive).

In order to apply the criterion (24), we need to know the coefficient κ tied to the value of the T_{I} stress generated by the internal pressure alone. The full solution to this elasticity problem is given in the book of Kassir and Sih (1975), Chap. 1. From there, one can derive the value of κ; the result is

$$\kappa=\tfrac{1}{2}+\nu \tag{25}$$

(see Leblond, 1993 for details).

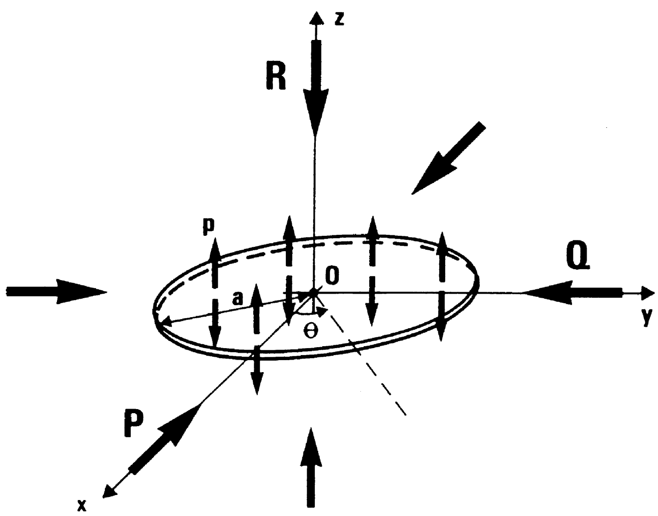

Fig. 7. Internally and externally loaded penny-shaped crack.

It remains to evaluate the non-singular stress $T_{\rm I}^0$ induced by the far stresses $-P$, $-Q$, $-R$ alone (the crack lips being traction-free). That due to $-P$ and $-Q$ is easily calculated; indeed imposing only these compressions results in a uniform stress tensor equal to $-P\mathbf{e}_x \otimes \mathbf{e}_x - Q\mathbf{e}_y \otimes \mathbf{e}_y$, so that the non-singular stress generated at the point θ of the crack front is $-P\cos^2\theta - Q\sin^2\theta$. To get that induced by $-R$, let us decompose the corresponding stress field in the following way:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^A + \boldsymbol{\sigma}^B, \quad \boldsymbol{\sigma}^A = \boldsymbol{\sigma} + R\mathbf{e}_z \otimes \mathbf{e}_z, \quad \boldsymbol{\sigma}^B = -R\mathbf{e}_z \otimes \mathbf{e}_z.$$

The stress field $\boldsymbol{\sigma}^A$ corresponds to a problem where the stress tensor vanishes at infinity and a tension R, or equivalently a negative pressure $-R$, is exerted on the crack lips; hence the non-singular stress generated is κR. The stress field $\boldsymbol{\sigma}^B$ does not create any non-singular stress. Therefore the non-singular stress due to $-R$ is κR, and that due to $-P$, $-Q$ and $-R$ is

$$T_{\rm I}^0 = -P\cos^2\theta - Q\sin^2\theta + \kappa R$$

at the point θ of the crack front.

It follows from this expression and eqn (25) that the stability criterion (24) reads

$$-P\cos^2\theta - Q\sin^2\theta + \kappa R + (1-\kappa)p = -P\cos^2\theta - Q\sin^2\theta + (\tfrac{1}{2}+\nu)R + (\tfrac{1}{2}-\nu)p < 0.$$

Now, propagation being supposed to be quasistatic, the pressure imposed on the crack lips cannot be arbitrary, but is determined by the condition that $K_{\rm I}$ be equal to the material toughness $K_{{\rm I}c}$. Since the classical expression of $K_{\rm I}$ for the problem considered is $K_{\rm I} = 2(p-R)\sqrt{a/\pi}$ where a denotes the radius of the crack (Kassir and Sih, 1975), this requires that p be given by

$$p = R + \frac{K_{{\rm I}c}}{2}\sqrt{\frac{\pi}{a}}.$$

(This is in fact the *minimum* pressure, corresponding to vanishing accelerations, that must be imposed in order to promote propagation.) Inserting this expression into the above stability criterion, one obtains the following final form of the latter:

$$-P\cos^2\theta - Q\sin^2\theta + R + \left(\frac{1}{2}-\nu\right)\frac{K_{{\rm I}c}}{2}\sqrt{\frac{\pi}{a}} < 0. \qquad (26)$$

Condition (26) allows for an easy discussion of crack path stability. Indeed, if R is greater than P or Q, $-P\cos^2\theta - Q\sin^2\theta + R$ is positive for $\theta = 0$ or $\pi/2$; since $(1/2-\nu)(K_{{\rm I}c}/2)\sqrt{\pi/a}$ is also positive, the stability criterion is violated at some locations of the crack front. Hence *crack orientations perpendicular to the major or intermediate* (in absolute value) *principal far stresses always lead to instability*. On the other hand, if R is smaller than P and Q, $-P\cos^2\theta - Q\sin^2\theta + R$ is smaller than $-\min(P,Q) + R$ and hence negative. The sign of the left-hand side of condition (26) depends on the magnitude of the principal far stresses with respect to $K_{{\rm I}c}/\sqrt{a}$. Since these stresses essentially depend upon the crack depth under the ground surface, it is concluded that *if the crack plane is orthogonal to the minor* (in absolute value) *principal stress*, *coplanar propagation is stable for depths greater than a certain critical value and unstable otherwise*.

It is an experimental fact that cracks induced by hydraulic fracturing are quite generally observed to develop orthogonally to the direction of minimum compression. This feature can of course be interpreted in a very simple way through considerations of crack *initiation*: indeed this direction is clearly favoured from that point of view since it corresponds to the minimum resistant forces that must be overcome by the pressure imposed in order to create a crack. The above discussion, however, provides another, complementary explanation: if a crack happens *not* to be generated perpendicularly to the direction of minimum compression (due for instance to a local toughness anisotropy), coplanar propagation is unstable and the crack orientation is bound to quickly change.

Exceptions to the experimental general rule just mentioned are sometimes observed to occur for very shallow depths. This may arise from the fact that the stresses due to gravity are very small for such depths as compared to those necessary to promote crack initiation, so that they might not induce any preferred crack orientation in that case; but the above discussion again yields a somewhat different and appealing interpretation. Indeed no crack orientation is preferred from the point of view of fracture path stability for depths smaller than the critical value, since all orientations then lead to instability.

Appendix

The aim of this Appendix is to provide expressions or numerical values for all non-zero components of the operators $\mathbf{F}(\varphi)$, $\mathbf{G}(\varphi)$ and $\mathbf{H}(\varphi)$. We use the notation

$$m \equiv \frac{\varphi}{\pi} \; (-1 < m < +1). \tag{A1}$$

The "in-plane" components of the operator $\mathbf{F}(\varphi)$ are given by the following formulae, the accuracy of which is better than 10^{-6} for $|\varphi| \leqslant 80^\circ$:

$$F_{\mathrm{I,I}}(\varphi) = 1-3.7011017m^2+6.064562m^4-6.337059m^6+5.07790m^8 \\ -2.88312m^{10}-0.0925m^{12}+2.996m^{14}-4.059m^{16}+1.63m^{18}+4.1m^{20}+O(m^{22}); \tag{A2}$$

$$F_{\mathrm{I,II}}(\varphi) = -4.7123890m+12.409868m^3-15.087784m^5+12.313906m^7 \\ -7.32433m^9+1.5793m^{11}+4.0216m^{13}-6.915m^{15}+4.21m^{17}+4.56m^{19}+O(m^{21}); \tag{A3}$$

$$F_{\mathrm{II,I}}(\varphi) = 1.5707963m-4.834754m^3+6.639793m^5-6.176023m^7 \\ +4.44112m^9-1.5340m^{11}-2.0700m^{13}+4.684m^{15}-3.95m^{17}-1.32m^{19}+O(m^{21}); \tag{A4}$$

$$F_{\mathrm{II,II}}(\varphi) = 1-7.7011017m^2+14.762653m^4-14.751719m^6+10.58254m^8 \\ -4.78511m^{10}-1.8804m^{12}+7.280m^{14}-7.592m^{16}+0.25m^{18}+12.5m^{20}+O(m^{22}). \tag{A5}$$

The "out-of-plane" component $F_{\mathrm{III,III}}(\varphi)$ admits the following simple analytic expression:

Table A1. Numerical values of the components of the operator $\mathbf{H}(\varphi)$

φ (°)	$H_{\mathrm{I,I}}(\varphi)$	$H_{\mathrm{I,II}}(\varphi)$	$H_{\mathrm{II,I}}(\varphi)$	$H_{\mathrm{II,II}}(\varphi)$
0	0	−2.250	0.750	0
5	−0.098	−2.236	0.746	−0.189
10	−0.194	−2.196	0.731	−0.374
15	−0.288	−2.129	0.708	−0.553
20	−0.377	−2.037	0.675	−0.723
25	−0.461	−1.922	0.635	−0.879
30	−0.538	−1.786	0.587	−1.021
35	−0.608	−1.631	0.533	−1.145
40	−0.669	−1.460	0.474	−1.250
45	−0.721	−1.276	0.410	−1.334
50	−0.763	−1.082	0.344	−1.396
55	−0.796	−0.881	0.276	−1.436
60	−0.819	−0.677	0.207	−1.454
65	−0.833	−0.472	0.139	−1.450
70	−0.837	−0.270	0.072	−1.424
75	−0.832	−0.073	0.008	−1.378
80	−0.818	0.115	−0.052	−1.313

$$F_{\mathrm{III,III}}(\varphi) = \left(\frac{1-m}{1+m}\right)^{m/2}. \tag{A6}$$

Similarly, the "in-plane" components of $\mathbf{G}(\varphi)$ are given by

$$G_{\mathrm{I,I}}(\varphi) = 15.7496106m^2 - 47.933390m^4 + 63.665987m^6 - 50.70880m^8 + 26.66807m^{10} - 6.0205m^{12} - 7.314m^{14} + 10.947m^{16} - 2.85m^{18} - 13.7m^{20} + O(m^{22}); \tag{A7}$$

$$G_{\mathrm{II,I}}(\varphi) = -5.0132566m + 30.079540m^3 - 59.565733m^5 + 61.174444m^7 - 39.90249m^9 + 15.6222m^{11} + 3.0343m^{13} - 12.781m^{15} + 9.69m^{17} + 6.62m^{19} + O(m^{21}) \tag{A8}$$

and the "out-of-plane" component $G_{\mathrm{III,II}}(\varphi)$ by

$$G_{\mathrm{III,II}}(\varphi) = -2m\sqrt{\frac{2\pi}{1-m^2}}\left(\frac{1-m}{1+m}\right)^{m}. \tag{A9}$$

Finally, numerical values of the "in-plane" components of $\mathbf{H}(\varphi)$ are provided in Table A1 for $0 \leqslant \varphi \leqslant 80°$ (values for $\varphi < 0$ are readily obtained from the easily proved fact that $H_{\mathrm{I,I}}$ and $H_{\mathrm{II,II}}$ are odd, and $H_{\mathrm{I,II}}$ and $H_{\mathrm{II,I}}$ even, functions of φ) whereas the "out-of-plane" component $H_{\mathrm{III,III}}(\varphi)$ is given by the following formula:

$$H_{\mathrm{III,III}}(\varphi) = \frac{1}{\cos(\pi m/2)}\left[\frac{3}{4}\left(\frac{1-m}{1+m}\right)^{m/2}\sin\left(\frac{\pi m}{2}\right) - \frac{2m}{\sqrt{1-m^2}}\left(\frac{1-m}{1+m}\right)^{m}\right]. \quad \text{(A10)}$$

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